Cheap Diagonalization and Fibonacci Numbers

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1 Introduction

I wanted to show my students how matrix theory can be used to generate explicit formulas for recursive sequences like the Fibonacci Numbers. Matrix diagonalization is one way to go about this, but I wanted to pursue a more calculational approach, which I detail below.

Students in my Linear Algebra class were familiar with stochastic matrices from a prior unit. Because they had already seen how A^n and $A^n v$ converge for a square stochastic matrix A and vector v, I started our investigation of Fibonacci Numbers by posing the following generalization.

Inquiry 1. *Let A be a 2×2 matrix and v a 2×1 vector. We investigate the effect of repeatedly multiplying v by A and set $v_n = A^n v$, so that $v_0 = v$. In this case, does the sequence of vectors $\{v_n\}_{n \geq 0}$ converge? If not, can we find a nonzero real number so that the sequence $\{\frac{1}{\alpha^n} v_n\}_{n \geq 0}$ converges to a nonzero vector $\bar{v}$?*

To give my students a little experience with the material, I gave them the following calculation.

Numberical Investigation 1. (**NI1**):

Set $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ and $v = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$. Then the first few terms of the sequence $\{v_n\}_{n \geq 0}$ are

$$v_0 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$
$$v_1 = \begin{pmatrix} 5 \\ 8 \end{pmatrix}$$
$$v_2 = \begin{pmatrix} 13 \\ 28 \end{pmatrix}$$
$$v_3 = \begin{pmatrix} 41 \\ 80 \end{pmatrix}$$
$$v_4 = \begin{pmatrix} 121 \\ 244 \end{pmatrix}.$$

Taking the ratio of successive corresponding terms gives

$$\begin{array}{ll} \frac{5}{1} = 5 & \frac{8}{4} = 2 \\ \frac{13}{5} = 2.6 & \frac{28}{8} = 3.5 \\ \frac{41}{13} = 3.2 & \frac{80}{28} = 2.9 \\ \frac{121}{41} = 3.0 & \frac{244}{80} = 3.1. \end{array}$$

suggesting that v_{n+1} is approximately 3 times v_n. Said another way, if $\alpha = 3$, the sequence $\{\frac{1}{\alpha^n} v_n\}_{n \geq 0}$ seems to converge. This leads us to a definition.

Definition 1. *Let A be a 2×2 matrix and v be a nonzero 2×1 vector. If α is a non-zero real number such that $\lim_{n\to\infty} \frac{1}{\alpha^n} v_n = \bar{v}$, a non-zero vector, then α is a* ***scale factor*** *of A for vector v, and $\bar{v}$ is the* ***scaled limit*** *of v with respect to A.*

Indeed, if α is a scale factor of A for vector v, then $\frac{1}{\alpha} A\bar{v} = \bar{v}$. Changing the name of our variables and rearranging leads to the famous equation

$$Ax = \lambda x.$$

Subtracting λx from both sides and noticing that $A - \lambda I$ must be singular leads to $\det(A - \lambda I) = 0$. The solutions of this equation are the eigenvalues λ_1 and λ_2 which have corresponding eigenvectors x_1 and x_2. (I restricted examples in the class to 2×2 matrices with two distinct eigenvalues.)

Using the matrix and vector from **NI 1** we find eigenvalues and eigenvectors $\lambda_1 = -1$, $x_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\lambda_2 = 3$, $x_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

To be able to say more about our sequence of vectors $\{v_n\}_{n\geq 0}$, we use these eigenvecors to find an explicit formula for v_n. Notice that since x_1 and x_2 are eigenvectors with distinct eigenvalues, they form a basis for $\mathbf{R}^2$: any vector v can be written as a linear combination of x_1 and x_2, $v = m_1x_1+m_2x_2$. Continuing **NI 1**, $v = -\frac{1}{2}x_1 + \frac{3}{2}x_2$, so that

$$\begin{aligned} A^n v &= -\frac{1}{2}A^n x_1 + \frac{3}{2}A^n x_2 \\ v_n &= -\frac{1}{2}(-1)^n x_1 + \frac{3}{2}3^n x_2. \end{aligned}$$

Rescaling v_n and taking the limit we see that

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{3^n} v_n &= \lim_{n\to\infty} -\frac{(-1)^n}{2\cdot 3^n} x_1 + \frac{3}{2}x_2 \\ &= 0 \cdot x_1 + \frac{3}{2}x_2 \\ &= \begin{pmatrix} 1.5 \\ 3 \end{pmatrix}. \end{aligned}$$

This means that $\{\frac{1}{3^n}v_n\}_{n\geq 0}$ converges, proving our suspicion that 3 is a scale factor of A for vector v.

A consequence of our investigation above was that we found an explicit formula for powers of a matrix A multiplied by a vector v. We turn these techniques to an application involving the Fibonacci Numbers. Students had previously been introduced to the Fibonacci Numbers and their recursive definition: $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$. Amazingly, this sequence has a realization in matrix terms. If matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, then for any vector $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, $A\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} f_1 + f_2 \\ f_1 \end{pmatrix}$, suggesting the Fibonacci recursive definition. Indeed, if $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$, then $A^n v = v_n = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$.

Calculation reveals the eigenvalues for A are $\lambda_1 = \frac{1+\sqrt{5}}{2}$ and $\lambda_2 = \frac{1-\sqrt{5}}{2}$, with eigenvectors $x_1 = \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}$ and $x_2 = \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$.

Writing v as a linear combination of x_1 and x_2 gives

$$v = \frac{1}{\sqrt{5}}x_1 - \frac{1}{\sqrt{5}}x_2.$$

and

$$v_n = \frac{1}{\sqrt{5}}{\lambda_1}^n x_1 - \frac{1}{\sqrt{5}}{\lambda_2}^n x_2.$$

Taking only the second element of vector v_n yields an explicit formula for the n^{th} Fibonacci number:

$$F_n = \frac{1}{\sqrt{5}}{\lambda_1}^n - \frac{1}{\sqrt{5}}{\lambda_2}^n.$$

Amazingly, even though $\sqrt{5}$ appears many times in this formula and in the work to derive it, F_n is always an integer. This formula allows us to directly calculate F_n without having to calculate all the preceding Fibonacci Numbers and is useful in further investigations.